

# Angular momentum and geometrical phases in tight-focused circularly polarized plane waves

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Calculations for the field at the focal plane of a high numerical aperture lens focusing a circularly polarized plane wave are presented. The calculations show that the polarization of the wave front in the focal plane is space varying, and that a geometrical phase is added to the wave front. Calculation of the angular momentum at the focal plane reveals that it depends on the numerical aperture of the lens. It is shown that this dependence is directly connected to the lens acting as a spatial filter.

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The focusing of light through high numerical aperture (NA) lenses has been the topic of much research.<sup>1-3</sup> Understanding how tightly focused beams propagate is important for a variety of applications such as microscopy<sup>3</sup> and optical tweezing.<sup>4</sup> A particularly interesting aspect of the studies is how optical beams carrying angular momentum<sup>5</sup> behave when they are focused through high NA lenses. Such beams have many practical applications as they can be used to apply torque to small particles, making them useful for studies in a variety of fields such as cell mechanics.<sup>6</sup>

In the paraxial regime the angular momentum of a beam is the sum of two components: orbital angular momentum  $l$ , which is associated with the spiral phase of the beam, and intrinsic angular momentum or helicity  $\sigma$ , which is associated with the polarization of the beam ( $\pm 1$  for circular polarization). Although much research has been done on how angular momentum is manifested in nonparaxial beams (see Ref. 5 for a review), little work has been done on the specific topic of angular momentum in tightly focused beams. This study attempts to shed light on how angular momentum is manifested in such beams.

According to the classical paper by Richards and Wolf<sup>7</sup> the electric field  $\mathbf{E}$  in the focal plane of an aplanatic lens, when a linearly polarized point source located at infinity is stigmatically imaged, by a lens with focal length  $f$  is

$$\mathbf{E} = \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \begin{bmatrix} -iA[I_0 + I_2 \cos(2\phi)] \\ -iAI_2 \sin(2\phi) \\ -2AI_1 \cos \phi \end{bmatrix}, \quad (1)$$

where

$$I_0(\rho) = \int_0^{k \sin \alpha} F_0[\kappa] J_0[\kappa \rho] \exp(i\sqrt{k^2 - \kappa^2} z) d\kappa, \quad (2a)$$

$$I_1(\rho) = \int_0^{k \sin \alpha} F_1[\kappa] J_1[\kappa \rho] \exp(i\sqrt{k^2 - \kappa^2} z) d\kappa, \quad (2b)$$

$$I_2(\rho) = \int_0^{\sin \alpha} F_2(\kappa) J_2[\kappa \rho] \exp(i\sqrt{k^2 - \kappa^2} z) d\kappa. \quad (2c)$$

In Eqs. (1) and (2)  $k$  represents the wave number,  $\sin \alpha$  is the NA,  $\kappa = k \sin \theta_0$  is the spatial frequency with  $\theta_0$  representing the angle between the corresponding rays in the image space and the optical axis.  $\rho, \phi, z$  represent coordinates in a cylindrical frame of reference originating at the focal spot, and with the  $z$  axis oriented along the optical axis. Furthermore, the exact forms of  $F_0-F_2$  can be found in Ref. 7. Figure 1 illustrates the geometry and the definitions mentioned above.

Circularly polarized light is simply the superposition of two orthogonally linearly polarized beams with a retardation of  $\pi/4$  between them. Thus the field in the image space when the incident beam is circularly polarized can be calculated by adding the field for an incident polarized in the  $x$  direction [Eq. (1)] with the field calculated for a beam polarized in the  $y$  direction, which is retarded by  $\pi/4$ .

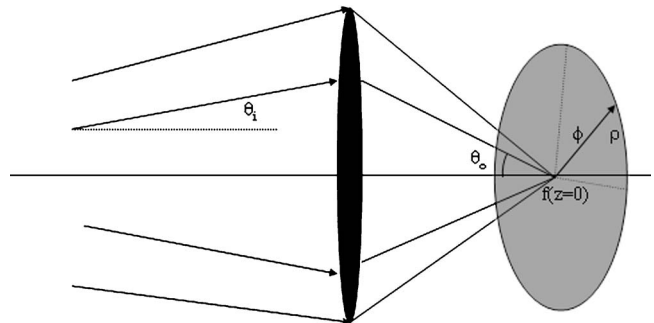


FIG. 1. Illustration showing the geometry of the system and definitions of the various coordinates.

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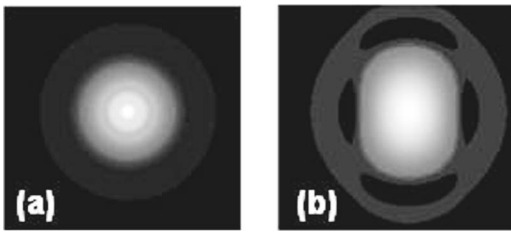


FIG. 2. (a) Electric energy at the focus of a lens with a NA of 0.95, when the incident beam is circularly polarized and (b) linearly polarized.

$$\mathbf{E}_{\text{circ}} = \begin{bmatrix} e_x + ie'_x \\ e_y + ie'_y \\ e_z + ie'_z \end{bmatrix} = \begin{bmatrix} -iA[I_0 + I_2 \exp(i2\phi)] \\ A[I_0 - I_2 \exp(i2\phi)] \\ -2AI_1 \exp(i\phi) \end{bmatrix}. \quad (3)$$

In order to derive Eq. (3) the field for the component polarized in the  $y$  direction was found by rotating the fields given in Eq. (1) by  $\pi/2$  and inserting into the solution  $\phi = \phi' - \pi/2$ , where  $\phi'$  is the azimuthal coordinate in the rotated field.

Figure 2(a) shows the electric energy density,  $w_e = (1/16\pi)(\mathbf{E} \cdot \mathbf{E}^*)$ , at the focus of a lens with a NA of 0.95 when the incident beam is circularly polarized. The energy distribution possesses a circular symmetry, in contrast to the asymmetric distribution observed when linearly polarized light is focused with this lens as shown in Fig. 2(b).

Focusing a beam through a high NA lens causes depolarization.<sup>8</sup> Therefore it is interesting to examine the polarization of the wave front in the focal plane. This can be done by calculating the Stokes parameters<sup>9</sup> of the field at each point, from which the azimuthal angle  $\psi$  and the ellipticity  $\tan \chi$  can be found.

$$\tan 2\psi = \tan[2\phi \operatorname{sgn}(I_0 I_2)], \quad (4a)$$

$$\sin 2\chi = (I_0 - I_2)/(I_0 + I_2). \quad (4b)$$

Figure 3 shows (a) a cross section of the ellipticity as well as (b) the azimuthal angle of the local polarization ellipses and (c) the intensity of the transverse electric field at the focus of a lens with NA=0.95. The polarization is space varying. Although the wave front at the focal point is circularly polarized, the ellipticity drops sharply close to the first intensity minima, where it changes from the original left hand polarization to right hand polarization. The azimuthal angle is either radially or azimuthally oriented, depending on the distance of the point at which it is measured from the center. In the main lobe the orientation is radial; however, as the ellipticity changes signs (this corresponds to a change in the handedness of the polarization), the azimuthal angle switches from radial to azimuthal. It should be noted that at the boundaries of these changes, the polarization is circular, and the azimuthal angle is undefined. These boundaries are in fact polarization singularities.<sup>10,11</sup>

The phase of a beam with space-variant polarization can be calculated using Pancharatnam's definition for phase,<sup>12</sup>  $\varphi_p = \arg\langle \mathbf{E}(\mathbf{r}_1), \mathbf{E}(\mathbf{r}_2) \rangle$ , where  $\langle \rangle$  denotes an inner product and  $\mathbf{E}(\mathbf{r}_1)$  and  $\mathbf{E}(\mathbf{r}_2)$  are the electric fields at two different points on the wave front. Calculating the phase between the transverse components of the wave front at two points located on the same circle around the focal spot yields

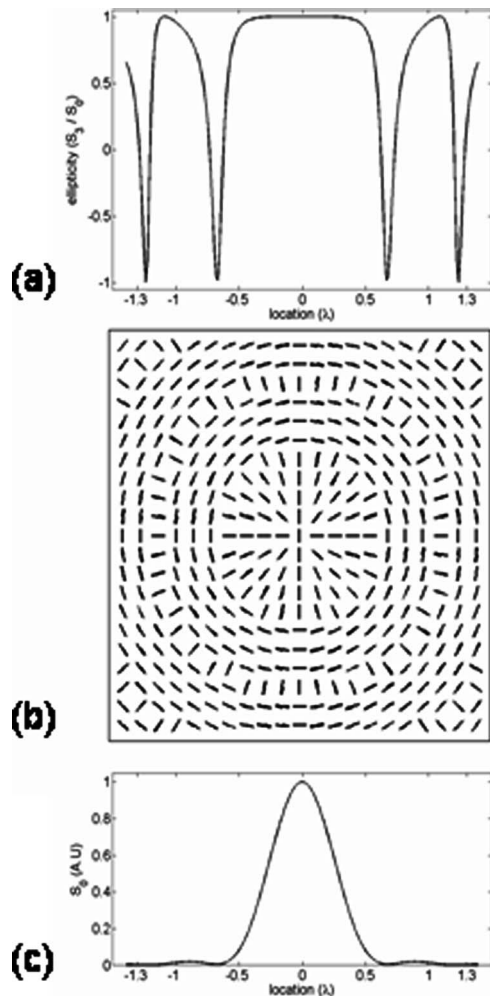


FIG. 3. Transverse polarization when a circularly polarized beam is focused through a lens with NA=0.95. (a) A cross section of the ellipticity of the beam, (b) the azimuthal angle, and (c) the intensity of the transverse field.

$$\varphi_p = \arg\langle \mathbf{E}(r, 0), \mathbf{E}(r, \phi) \rangle = \phi - a \tan \left[ \frac{I_0^2 - I_2^2}{I_0^2 + I_2^2} \tan \phi \right], \quad (5)$$

which, by application of Eq. (4b), yields,

$$\varphi_p = \phi - a \tan[(\sin 2\chi) \tan \phi], \quad (6)$$

where  $\chi$  is the ellipticity of  $\mathbf{E}(r, 0)$  and  $\mathbf{E}(r, \phi)$ . Thus, the beam seems to possess a spiral phase, which suggests that it possesses orbital angular momentum. However, the paraxial definition of orbital angular momentum does not hold, because the value of  $l$  depends on the closed circuit along which integration is performed. Hence, the distinction between intrinsic angular momentum is blurred as expected in the nonparaxial regime.<sup>5,13</sup> Note that both  $\mathbf{E}(r, 0)$  and  $\mathbf{E}(r, \phi)$  have the same ellipticity as they are both situated on the same circle centered around the focal point [see Fig. 3(a)].

The distinction between orbital angular momentum and intrinsic angular momentum can be retrieved to some extent if the beam in the focal plane is decomposed into two components, each possessing uniform circular polarization of opposite helicity, so that Eq. (2a) takes on the form

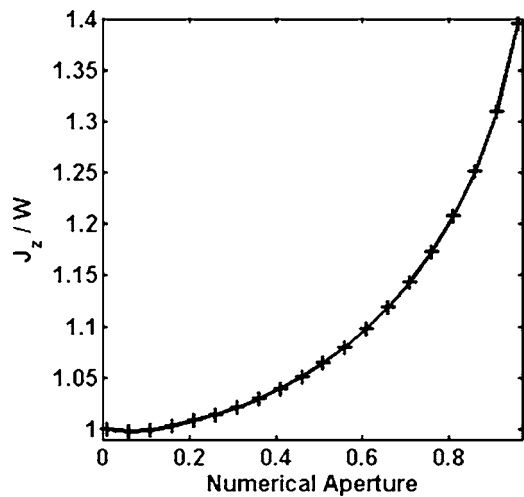


FIG. 4. Angular momentum-energy ratio in the focal plane of the lens as a function of its NA.

$$\mathbf{E} = -iA \left( \begin{bmatrix} 1 \\ i \\ e_{z1} \end{bmatrix} I_0(\kappa) + \begin{bmatrix} 1 \\ -i \\ e_{z2} \end{bmatrix} I_2(\kappa) \exp(i2\phi) \right), \quad (7)$$

where  $e_{z1}$  and  $e_{z2}$  are the respective components in the  $z$  direction. The first component in Eq. (7) has helicity of  $\sigma=1$  and orbital angular momentum  $l=0$ , whereas the second component has helicity  $\sigma=-1$  and orbital angular momentum  $l=2$ . The first component has the same polarization as the beam in the object space, and has undergone no phase modification. On the other hand, the polarization of the second component has been reversed, and it has gained a spiral phase with topological charge  $l=2$ . In the paraxial approximation, the angular momentum of a beam per unit energy is proportional to  $l+\sigma$ .<sup>5</sup> For each of the components in Eq. (7)  $l+\sigma=1$ , which is equal to the value of  $l+\sigma$  of the incident circularly polarized beam. This suggests that through focusing helicity has been converted into orbital angular momentum.

By Fermat's principle, the phase between the circularly polarized plane wave at the entrance pupil to the focal plane along all rays must be the same. Thus, the nonuniform phase in the focal plane cannot result from the propagation of the beam, and must be a geometrical phase that results from the change in polarization that occurs at each point of the beam.<sup>14</sup> In fact it is possible to show that  $\varphi_p$ , as calculated in Eq. (5), is equal to half the area of the geodesic triangle on the Poincare sphere<sup>9</sup> defined by the pole (circular polarization),  $\mathbf{E}(r, 0)$ , and  $\mathbf{E}(r, \theta)$ . This suggests a strong connection to the geometrical phase that is added to a beam when its polarization traverses a closed loop on the Poincare sphere,<sup>9</sup> indicating a strong connection between angular momentum and geometrical phases as suggested in the past.<sup>15</sup>

As a final point of discussion, we examine the ratio between the total angular momentum along the optical axis  $J_z$  and the energy of the beam  $W$ . To this end, the formula suggested by Barnett and Allen for the ratio between angular momentum and total energy of a nonparaxial beam is utilized,<sup>13</sup>

$$\frac{J_z}{W} = \frac{(l+\sigma)}{\omega} + \frac{\sigma}{\omega} \frac{\int_0^k d\kappa [E(\kappa)]^2 \kappa / (k^2 - \kappa^2)}{\int_0^k d\kappa (|E(\kappa)|^2 (2k^2 - \kappa^2) / \kappa (k^2 - \kappa^2))}, \quad (8)$$

where  $W$  and  $J_z$  are the total energy and angular momentum along the optical axis, respectively and  $\omega$  is the frequency of the wave. Equation (8) holds for beams with uniform polarization in the transverse field. Therefore, in order to obtain the ratio between the angular momentum and energy of the entire beam, Eq. (8) is applied to each of the components in Eq. (7) separately and then summed to yield Figure 4 shows calculations of the total angular momentum-energy ratio ( $J_z/W$ ) in the image space as a function of the NA of the lens. The ratio is almost constant when the NA is smaller than 0.2, after which it monotonically increases until a value of about 1.4 when the NA is 0.95. This presents an apparent paradox, as the lens is invariant under rotation, and hence the symmetry of the problem dictates the conservation of angular momentum.

To resolve this paradox we examine the angular momentum of the field at the front aperture of the lens, assuming that the point source is located at a distance  $S_1$  from the lens along the optical axis. The field at the front aperture can be found by substituting  $z=-S_1$ , and  $\kappa=k \sin \theta_i$ , where  $\theta_i$  is the angle between rays in the object space and the  $z$  axis, into Eqs. (1) and (2). Consequently, the angular momentum of the beam in the object space is given by Eq. (8), and is equal to the angular momentum of the beam in the image space.

This analysis shows that the NA-dependent increase in angular momentum of the focused beam shown in Fig. 4 is due to the lens acting as a low pass filter. The lens only transmits spatial frequencies lower than  $\kappa_{\text{cutoff}}=k(R/\sqrt{s_1^2+R^2})$ , where  $R$  is the aperture of the lens. If we assume that the focal length is kept constant, then  $\kappa_{\text{cutoff}}$  increases with NA. By Eq. (8), the high spatial frequencies carry more angular momentum per unit energy than the low spatial frequencies. Therefore increasing the aperture and hence the NA will inevitably increase the amount of AM per unit energy present in the focal region.

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